AIAA 80-0700R

Free and Forced Vibration of Closely Coupled Turbomachinery Blades

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This paper deals with the free and forced vibration of closely coupled turbomachinery blades on a disk which are connected by elements forming a circumferentially continuous periodic structure that will be called a rotationally periodic structure in this paper. A computational procedure for calculating the free vibration of rotationally periodic structures with various types of connecting elements is obtained by a transfer matrix method. The method of analysis for a packet formed by a finite number of blades is also discussed. The forced vibration and the condition of resonance of a rotationally periodic structure under a time-varying periodic excitation are studied, and a vibration design criterion is suggested.

Nomenclature

$[A],[B],[C],[D]$	= submatrices of the matrix $[N]$
$[B]$	= field transfer matrix of a connecting element
$[E],[F],[G],[H]$	= submatrices of the matrix $[B]$
$\{F_j\}$	= force vector at cross-section j of a blade
i	$=\sqrt{-1}$
$[I]$	= identity matrix
$[K]$	= stiffness matrix
K_{ij}	= elements of $[K]$
$[L]$	= flexibility matrix
L_{ij}	= elements of $[L]$
m	= number of nodal diameters of the mode pattern
$[M]$	= transfer matrix of the structure
M_x,M_y,M_z	= moment components
N	= total number of blades on the disk
n	= total number of segments divided in a blade
$[N]$	= point transfer matrix of a blade
$p.t$	= number of blades in a packet
$\{q\}$	= generalized displacement vector
$\{Q\}$	= generalized force vector
R,S,T,M,N	= nondimensional generalized forces defined in Eq. (23c)
U,V,F,L,K	= nondimensional generalized displacements defined in Eq. (23b)
t_c	= pitch of blades
u,v,w	= linear displacement components
V_x,V_y,V_z	= force components
x,y,z	= coordinates
$\{X\}$	= state vector
δ	= flexibility
θ	= angle on a disk measured from a reference radius
$\theta_x,\theta_y,\theta_z$	= angular displacement components
φ	$=2\pi/N$
ω_m	= natural angular frequency of the structure corresponding to the mode pattern with m nodal diameters
Superscripts	
r	= right-hand side
l	= left-hand side
Subscripts	
i	= number of blade in the structure
j	= number of cross section in a blade
k	= number of harmonic of the exciting force

Presented as Paper 80-0700 at the AIAA/ASME/ASCE/AHS 21st Structures, Structural Dynamics and Materials Conference, Seattle, Wash., May 12-14, 1980; submitted May 29, 1980; revision received Jan. 6, 1981.

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Introduction

THE blades of a disk on the exhaust stage of a turbine are usually connected in packets by elements such as lacing wires and shrouds. Recently, the loose lacing wires and the so-called Z-type lacing wires are sometimes used as connecting elements so that the blades on the disk are connected in one circumferentially closed structure. This closed blading structure has certain advantages in avoiding blade resonance, but the frequency spectrum and the vibration modes of such a structure are rather complex and hence an appropriate method for calculation of vibration is needed.

From the standpoint of mechanics such a structure is a rotationally periodic structure, the essential elements of which are a set of long twisted blades with variable cross sections of complex form, connected by elements of various types. For individual blades various methods of calculating vibration have been developed by many authors, for example, Jerrett and Werner.[1] Montoya[2] also developed a set of differential equations for coupled bending and torsional vibration of a twisted blade. This set of equations is used in this paper for the calculation of the dynamic stiffness matrix of a blade. For connected blades, Prohl[3] suggested a method of calculating the free vibration of a banded group of turbine buckets. Deak and Baird[4] developed a procedure for calculating the packet frequencies of turbine blades. In their method the coupling of bending and torsion of a blade is not considered, and the order of frequency determinant is relatively high. As for the rotationally periodic structure, Smolnickov[5] developed a general method for calculating its free vibration.

In this paper, the method of calculating the free vibration of circumferentially closed blading as well as discrete packets of blades is presented. In the calculation, the transfer matrix method is utilized. In addition, the forced vibration and condition of resonance of a closed structure of blades on a disk under time-periodic excitation are discussed, and a vibration design criterion of blades is suggested.

The Transfer Matrix Method for Calculating Rotationally Periodic Structures

Mechanical Model

Suppose that on a disk there are N equally spaced blades with similar vibrational characteristics (see Fig. 1). The numbers of blades are designated by $i(i=1, 2,..., N)$. The blades are connected to each other by similar connecting elements.

The long blades on the disk are beam-like structures. Let us examine one blade, say blade i, and divide it into n segments. The numbers of the cross sections of these segments are designated by j $(j=0, 1,...,n)$.

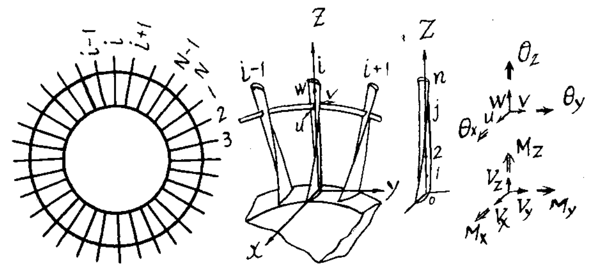

Fig. 1 Mechanical model.

The linear and the angular displacements of the structure are designated by $u, v, w, \theta_x, \theta_y, \theta_z$, and the corresponding forces and moments by $V_x, V_y, V_z, M_x, M_y, M_z$. Because the radial displacement of the blade is negligible, we may put $w = 0$ and do not have to consider the force V_z. Thus there are five generalized displacements and forces which are designated by $\{q\}$ and $\{Q\}$, respectively.

The Point Transfer Matrix

Consider a nodal point on the blade i to which the connecting element is attached. The reactions from right-hand side to the nodal point are designated by $\{Q\}_i^r$, and from left-hand side $-\{Q\}_i^l$. The reactions from the blade to the nodal point are equal to $[K]\ \{q\}$, where $[K]$ is the dynamic stiffness matrix of the blade. Thus the condition of continuity of the displacements and the condition of the equilibrium of forces at the nodal point may be written in the following matrix form:

$$\left\{\begin{array}{c} q \\ \hline Q \end{array}\right\}_i^r = \left[\begin{array}{c|c} I & 0 \\ \hline K & I \end{array}\right] \left\{\begin{array}{c} q \\ \hline Q \end{array}\right\}_i^l \tag{1}$$

Let the state vector $\{X\}$ and the point transfer matrix $[N]$ be

$$\{X\} = \left\{\begin{array}{c} q \\ \hline Q \end{array}\right\}; \quad [N] = \left[\begin{array}{c|c} I & 0 \\ \hline K & I \end{array}\right] \tag{2}$$

We have the nodal matrix equation

$$\{X\}_i^r = [N]\{X\}_i^l \tag{3}$$

Dynamic Stiffness Matrix of a Blade

The turbine blades of an exhaust stage are usually twisted with variable cross sections of complex form. Therefore the bending and torsional vibrations of a blade are coupled. The vibration motion of a blade can be described by Montoya's set of differential equations which takes the form [2]

$$\frac{\mathrm{d}}{\mathrm{d}t}\{X\} = [S]\{X\} \tag{4}$$

The dynamic stiffness matrix $[K]$ of the blade may be obtained by integrating the above differential equations. The integration results in a relation between the state vector of cross-section j_1 and that of cross-section j_2

$$\{X\}_{j_2} = [R]_{j_1, j_2}\{X\}_{j_1} \tag{5}$$

where the elements of matrix $[R]$ may be calculated as follows. At first, extend the matrix differential equation (4) to the form

$$\frac{\mathrm{d}}{\mathrm{d}t}[X] = [S][X] \tag{6}$$

where $[X]$ is a square matrix. Then take an identity matrix $[I]$ as initial values of cross-section j_1, that is, put $[X]_{j_1} = [I]$. Integrate Eq. (6) by numerical integration such as a Runge-Kutta method step by step from cross-sections j_1 to j_2 to obtain $[X]_{j_2}$. The $[X]_{j_2}$ thus obtained is

$$[X]_{j_2} = [R]_{j_1, j_2}[I] = [R]_{j_1, j_2} \tag{7}$$

and thus the matrix $[R]_{j_1, j_2}$ is obtained.

To derive the formula for the dynamic stiffness matrix $[K]$, suppose that the blade under study is clamped at the root and free at the tip. The end conditions of the blade are

$$\{q\}_0 = \{0\}; \quad \{Q\}_n = \{0\} \tag{8}$$

Suppose that connecting elements are attached to the blade at cross-section j, so that we may suppose that there are external forces $\{F_j\}$ acting at that cross section. Splitting the matrices in Eq. (5) into submatrices and substituting the end conditions of Eq. (8) into Eq. (5), we have

$$\left\{\begin{array}{c} q \\ \hline Q \end{array}\right\}_j = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]_{0,j} \left\{\begin{array}{c} 0 \\ \hline Q \end{array}\right\}_0 + \left\{\begin{array}{c} 0 \\ \hline -F_j \end{array}\right\} \tag{9}$$

and

$$\left\{\begin{array}{c} q \\ \hline 0 \end{array}\right\}_n = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]_{0,n} \left\{\begin{array}{c} 0 \\ \hline Q \end{array}\right\}_0 + \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]_{j,n} \left\{\begin{array}{c} 0 \\ \hline -F_j \end{array}\right\} \tag{10}$$

Expanding these matrix equations and solving for $\{q\}_j$,

$$\{q\}_j = [L]\{F_j\} = [B]_{0,j}[D]_{0,n}^{-1}[D]_{j,n}\{F\}_j \tag{11}$$

where $[L]$ is the dynamic flexibility matrix whose inverse is the dynamic stiffness matrix $[K]$. The elements of both matrices are functions of frequency ω. Substituting matrix $[K]$ into Eq. (2), the point transfer matrix $[N]$ may be obtained.

The Field Transfer Matrix

The field transfer matrix $[B]$ is the transfer matrix of connecting elements. The relation is

$$\{X\}_i^l = [B]\{X\}_{i-1}^r \tag{12}$$

The expressions of matrix $[B]$ are different for different types of connecting elements.

The Equation of Motion and Its Solutions

Combining Eqs. (3) and (12), we have the transfer matrix of the structure

$$[M] = [N][B] \tag{13}$$

and the relation of state vectors is

$$\{X\}_i^r = [M]\{X\}_{i-1}^r \tag{14}$$

Since the structure is periodic and circumferentially closed, after N transformations and multiplications we return to the initial blade, so that the equation of motion for the structure has the form

$$([M]^N - [I])\{X\} = \{0\} \tag{15}$$

The frequency determinant is

$$|[M]^N - [I]| = 0 \tag{16}$$

For convenience of solution the identity matrix $[I]$ is written in its complex form [5]

$$[I] = [I]e^{i2\pi m} \quad (m = 0, 1, 2, \ldots) \tag{17}$$

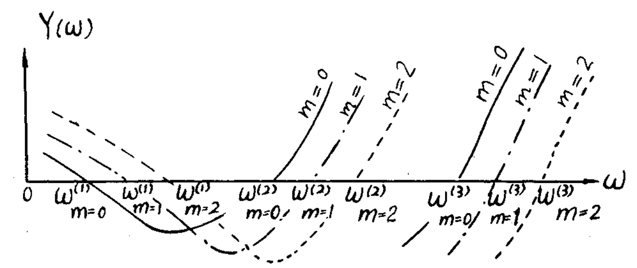

Fig. 2 Curves for determining frequencies.

where m is an integer indicating the number of nodal diameters in the pattern of vibration modes.

Designating

$$\varphi = 2\pi/N \tag{18}$$

then from Eq. (16), we obtain a series of frequency determinants

$$|[M] - [I]e^{im\varphi}| = 0 \tag{19}$$

where

$$m = 0,1,2,\ldots,(N/2) - 1 \qquad \text{(for even } N\text{)}$$

$$m = 0,1,2,\ldots,(N-1)/2 \qquad \text{(for odd } N\text{)}$$

Frequency determinants of Eq. (19) are complex. Expanding the determinants and separating the real and imaginary parts, we have

$$Y(\omega) + iZ(\omega) = 0 \tag{20}$$

or

$$Y(\omega) = 0 \text{ or } Z(\omega) = 0 \tag{21}$$

For every m and for a series of values of ω, we can calculate the function $Y(\omega)$ and plot the curves Y-ω. The intersections of these curves with the abscissa give frequencies for various vibration modes (see Fig. 2).

Some Practical Applications

The method described in the previous section has been used to calculate vibration frequencies of blades of a 600 mW turbine, a 100 mW turbine, and other rotationally periodic structures. Here we give some simple examples to show how the method is applied to practical structures.

Circumferentially Closed Structure of Blades Connected by Loose Lacing Wires

Suppose that the lacing wire is loosely attached to cross-section j of the blades (see Fig. 3). The lacing wire is considered as a continuous beam. When the disk rotates, the centrifugal force produces a friction which prevents relative sliding motion between blade and lacing wire. Thus the connection between blade and lacing wire may be considered as hinged, so that there is no moment at the points of connection:

$$M_x = M_y = M_z = 0 \tag{22}$$

In the following, we introduce a state vector in its non-dimensional form:

$$\{X\} = \begin{Bmatrix} q \\ \hline Q \end{Bmatrix} \tag{23a}$$

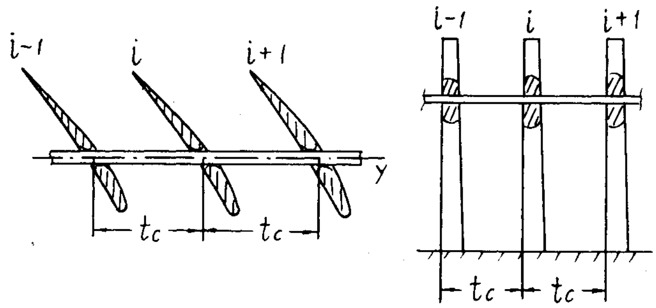

Fig. 3 Loose wire connection.

where

$$\{q\} = \begin{Bmatrix} U \\ V \\ F \\ L \\ K \end{Bmatrix} = \begin{Bmatrix} u/l \\ v/l \\ \theta_z \\ -\theta_x \\ \theta_y \end{Bmatrix} \tag{23b}$$

$$\{Q\} = \begin{Bmatrix} R \\ S \\ T \\ M \\ N \end{Bmatrix} = \frac{1}{EI_0} \begin{Bmatrix} lV_x \\ lV_y \\ M_z \\ M_x \\ M_y \end{Bmatrix} \tag{23c}$$

where l and EI_0 are characteristic parameters of the blade. From Eq. (22) it follows that

$$M_j = N_j = T_j = 0 \tag{24}$$

Equation (11) relating the dynamic flexibility matrix of a blade becomes

$$\{q\}_j = \begin{Bmatrix} U \\ V \\ F \\ L \\ K \end{Bmatrix}_j = [L]\{F\}_j$$

$$= \begin{bmatrix} L_{11} & L_{12} & * & * & * \\ L_{21} & L_{22} & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \cdot \begin{Bmatrix} R_j \\ S_j \\ T_j(=0) \\ M_j(=0) \\ N_j(=0) \end{Bmatrix} \tag{25}$$

where the symbol (*) denotes the elements that need not be calculated. It means that Eq. (25) can be reduced to

$$\begin{Bmatrix} U \\ V \end{Bmatrix}_j = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{Bmatrix} R \\ S \end{Bmatrix}_j \tag{26}$$

Then we can obtain the matrix of dynamic stiffness of the

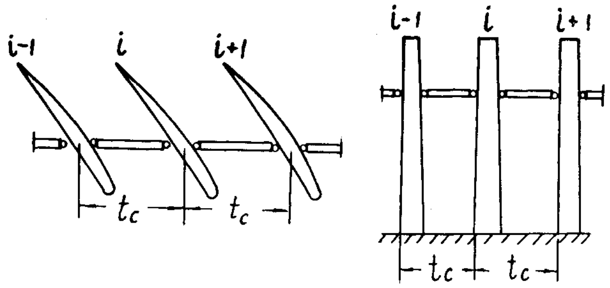

Fig. 4 Hinged wire connection.

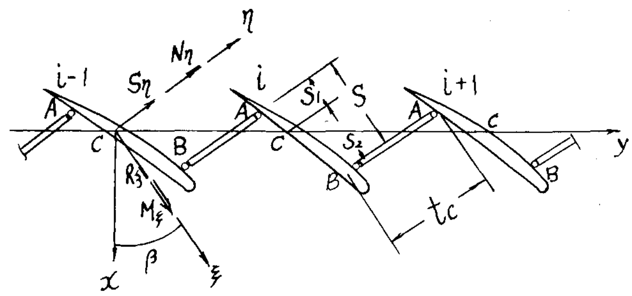

Fig. 5 Loose Z-type wire connections.

blade by the inverse of the matrix of flexibility

$$\begin{Bmatrix} R \\ S \end{Bmatrix}_j = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}^{-1} \begin{Bmatrix} U \\ V \end{Bmatrix}_j = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} U \\ V \end{Bmatrix}_j \tag{27}$$

Substituting Eq. (27) into Eq. (2) we can obtain the point transfer matrix of the blade, and the frequency spectrum and the vibration modes can be solved by way of the method described in the previous section.

Circumferentially Closed Structure of Blades Connected by Hinged Wires

If we neglect resistance of the wire to bending in the previous example, then the lacing wire may be considered as segments of wires with hinged ends (see Fig. 4). Equation (25) becomes

$$\begin{Bmatrix} U \\ V \\ F \\ L \\ K \end{Bmatrix}_j = \begin{bmatrix} * & * & * & * & * \\ * & L_{22} & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \begin{Bmatrix} R_j\ (=0) \\ S_j \\ T_j\ (=0) \\ M_j\ (=0) \\ N_j\ (=0) \end{Bmatrix} \tag{28}$$

And the point transfer matrix is reduced to

$$\begin{Bmatrix} V \\ S \end{Bmatrix}_i^r = [N] \begin{Bmatrix} V \\ S \end{Bmatrix}_i^l = \begin{bmatrix} 1 & 0 \\ (1/L_{22}) & 1 \end{bmatrix} \begin{Bmatrix} V \\ S \end{Bmatrix}_i^l \tag{29}$$

The field transfer matrix is reduced to

$$\begin{Bmatrix} V \\ S \end{Bmatrix}_i^l = [B] \begin{Bmatrix} V \\ S \end{Bmatrix}_{i-1}^r = \begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} V \\ S \end{Bmatrix}_{i-1}^r \tag{30}$$

where $\delta = (t_c/EF_c)\ (EI_0/l^3)$ is the tensile flexibility of the wire.

The transfer matrix of structure is

$$[M] = [N][B] = \begin{bmatrix} 1 & \delta \\ (1/L_{22}) & 1 + (\delta/L_{22}) \end{bmatrix}$$

The frequency determinant of Eq. (19) becomes

$$|[M] - [I]e^{im\varphi}| = \begin{vmatrix} 1 - e^{im\varphi} & \delta \\ (1/L_{22}) & 1 + (\delta/L_{22}) - e^{im\varphi} \end{vmatrix}$$

$$= e^{i2m\varphi} - (2 + \delta/L_{22})e^{im\varphi} + 1 = 0 \tag{31}$$

The frequency equation is

$$\delta/L_{22} = -4\sin^2(m\pi/N) \tag{32}$$

Borishansky[6] derived the frequency equation of a circumferentially closed structure composed of uniform rods connected by springs. He solved this problem by means of equations of finite differences. His result may be considered as a special case of Eq. (32).

Circumferentially Closed Structure of Blades Connected by Loose Z-Type Wires

The diagram of the structure is shown in Fig. 5. C_{xy} is the coordinate system of the blade and $C_{\xi\eta}$ the coordinate system of the wire, where C is the center of rigidity of the blade.

In the coordinate system of blade C_{xy}:

$$M_x = M_y = 0, \quad \text{or } M_j = N_j = 0 \tag{33}$$

In the coordinate system of wire $C_{\xi\eta}$

$$R_\xi = M_\xi = N_\eta = 0 \tag{34}$$

In the coordinate system of blade C_{xy}, Eq. (11) becomes

$$\{q\}_j = \begin{Bmatrix} U \\ V \\ F \\ L \\ K \end{Bmatrix}_j = \begin{bmatrix} L_{11} & L_{12} & L_{13} & * & * \\ L_{21} & L_{22} & L_{23} & * & * \\ L_{31} & L_{32} & L_{33} & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \begin{Bmatrix} R_j \\ S_j \\ T_j \\ M_j\ (=0) \\ N_j\ (=0) \end{Bmatrix} \tag{35}$$

For the transformation of coordinates from C_{xy} to $C_{\xi\eta}$, we introduce a coordinate transformation matrix $[\lambda]$ as

$$[\lambda] = \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{36}$$

Then for the center of rigidity C of a blade it gives

$$\begin{Bmatrix} U_\xi \\ V_\eta \\ F \end{Bmatrix}_C = [\lambda]^T \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} [\lambda] \begin{Bmatrix} R_\xi\,(=0) \\ S_\eta \\ T \end{Bmatrix}_C$$

$$= \begin{bmatrix} * & * & * \\ * & l_{22} & l_{23} \\ * & l_{32} & l_{33} \end{bmatrix} \begin{Bmatrix} R_\xi\,(=0) \\ S_\eta \\ T \end{Bmatrix}_C \tag{37}$$

so that Eq. (37) reduces to

$$\begin{Bmatrix} V_\eta \\ F \end{Bmatrix}_C = \begin{bmatrix} l_{22} & l_{23} \\ l_{32} & l_{33} \end{bmatrix} \begin{Bmatrix} S_\eta \\ T \end{Bmatrix}_C \tag{38}$$

For the transformation of displacements and forces from center of rigidity C to points of attachment A and B of Z-type wire, considering the equivalence of displacements and forces in plane motion, we have

$$\begin{Bmatrix} V_\eta \\ F \end{Bmatrix}_C = \frac{1}{\bar{s}} \begin{bmatrix} \bar{s}_2 & \bar{s}_1 \\ 1 & -1 \end{bmatrix} \begin{Bmatrix} V^r_{\eta B} \\ V^l_{\eta A} \end{Bmatrix}$$

$$\begin{Bmatrix} S_\eta \\ T \end{Bmatrix}_C = \begin{bmatrix} 1 & -1 \\ \bar{s}_1 & \bar{s}_2 \end{bmatrix} \begin{Bmatrix} S^r_{\eta B} \\ S^l_{\eta A} \end{Bmatrix} \tag{39}$$

where s_1 and s_2 are the distances shown in Fig. 5, and

$$\bar{s}_1 = s_1/l; \quad \bar{s}_2 = s_2/l; \quad \bar{s} = \bar{s}_1 + \bar{s}_2$$

l is the characteristic length. Substituting Eq. (39) into Eq. (38), we have

$$\begin{Bmatrix} V_\eta \\ S_\eta \end{Bmatrix}^r_B = [N] \begin{Bmatrix} V_\eta \\ S_\eta \end{Bmatrix}^l_A = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{Bmatrix} V_\eta \\ S_\eta \end{Bmatrix}^l_A \tag{40}$$

where

$$N_{11} = (l_{22} + \bar{s}_1(l_{32} + l_{23}) + \bar{s}_1^2 l_{33})/\Delta$$

$$N_{22} = (l_{22} - \bar{s}_2(l_{32} + l_{23}) + \bar{s}_2^2 l_{33})/\Delta$$

$$N_{12} = \bar{s}^2(l_{22}l_{33} - l_{23}l_{32})/\Delta$$

$$N_{21} = 1/\Delta$$

$$\Delta = l_{22} + \bar{s}_1 l_{23} - \bar{s}_2 l_{32} - \bar{s}_1\bar{s}_2 l_{33}$$

The field transfer matrix is similar to that in Eq. (30)

$$\begin{Bmatrix} V_\eta \\ S_\eta \end{Bmatrix}^l_{A,i} = [B] \begin{Bmatrix} V_\eta \\ S_\eta \end{Bmatrix}^r_{B,i-1} = \begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} V_\eta \\ S_\eta \end{Bmatrix}^r_{B,i-1} \tag{41}$$

Finally the frequency equation is obtained as follows

$$\frac{\delta + \bar{s}^2 l_{33} + \bar{s}(l_{32} + l_{23})}{l_{22} + \bar{s}_1 l_{23} - \bar{s}_2 l_{32} - \bar{s}_1\bar{s}_2 l_{33}} = -4\sin^2\frac{m\pi}{N} \tag{42}$$

Cross-Closed Packets of Blades Connected by Loose Wires

The structure is shown in Fig. 6 with the notation involved. The wires are considered as hinged segments.

Since there are two rows of wires attached to cross-sections i and j of the blades, respectively, the matrix of dynamic stiffness is extended from Eq. (28) to the following form:

$$[K]_{2\times 2} = \begin{bmatrix} L^{ii}_{22} & L^{ij}_{22} \\ L^{ji}_{22} & L^{jj}_{22} \end{bmatrix}^{-1} \tag{43}$$

and the point transfer matrix is then represented as

$$[N]_{4\times 4} = \left[\begin{array}{c|c} I & 0 \\ \hline K & I \end{array}\right] \tag{44}$$

The field transfer matrix is extended from Eq. (30) to

$$[B]_{4\times 4} = \left[\begin{array}{cc|cc} 1 & & \delta_i & \\ & 1 & & \delta_j \\ \hline 0 & & 1 & \\ & & & 1 \end{array}\right] \tag{45}$$

Then the transfer matrix of structure may be calculated by

$$[M]_{4\times 4} = [N][B] \tag{46}$$

The end conditions are

$$(S_i)^l_{p0+1} = 0; \quad (S_j)^r_{p0+t} = 0 \tag{47}$$

where S is defined by Eq. (23c).

From the recurrent relation it gives

$$\begin{Bmatrix} V_i \\ V_j \\ S_i \\ 0 \end{Bmatrix}^r_{p0+t} = [M]^{t-1} \left[\begin{array}{c|c} I & 0 \\ \hline K & I \end{array}\right] \begin{Bmatrix} V_i \\ V_j \\ 0 \\ S_j \end{Bmatrix}^l_{p0+1}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{Bmatrix} V_i \\ V_j \\ 0 \\ S_j \end{Bmatrix}^l_{p0+1} \tag{48}$$

From the fourth equation of Eq. (48) we solve for $(V_i)^l_{p0+1}$

$$(V_i)^l_{p0+1} = -\frac{a_{42}}{a_{41}}(V_j)^l_{p0+1} - \frac{a_{44}}{a_{41}}(S_j)^l_{p0+1}$$

Substituting this into Eq. (48) leads to

$$\begin{Bmatrix} V_i \\ S_i \end{Bmatrix}^r_{p0+1} = \begin{bmatrix} a_{12} - \dfrac{a_{11}a_{42}}{a_{41}}, & a_{14} - \dfrac{a_{11}a_{44}}{a_{41}} \\ a_{32} - \dfrac{a_{31}a_{42}}{a_{41}}, & a_{34} - \dfrac{a_{31}a_{44}}{a_{41}} \end{bmatrix} \begin{Bmatrix} V_j \\ S_j \end{Bmatrix}^l_{p0+1}$$

$$= [A] \begin{Bmatrix} V_j \\ S_j \end{Bmatrix}^l_{p0+1} = [A] \begin{bmatrix} 1 & \delta_j \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} V_j \\ S_j \end{Bmatrix}^r_{p0} \tag{49}$$

The additional end conditions are

$$(S_j)^l_{p0+t+1} = 0; \quad (S_i)^r_{p0+p} = 0 \tag{50}$$

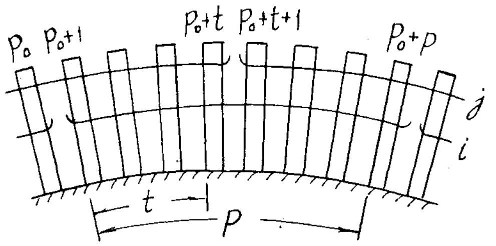

Fig. 6 Cross-closed connections.

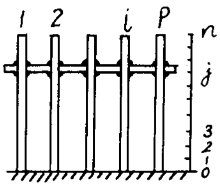

Fig. 7 Finite packet of blades.

From the recurrent relation

$$\begin{Bmatrix} V_i \\ V_j \\ 0 \\ S_j \end{Bmatrix}^r_{p0+p} = [M]^{p-t-1} \left[\begin{array}{c|c} I & 0 \\ \hline K & I \end{array}\right] \begin{Bmatrix} V_i \\ V_j \\ S_i \\ 0 \end{Bmatrix}^l_{p0+t+1}$$

$$= \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} \begin{Bmatrix} V_j \\ V_j \\ S_i \\ 0 \end{Bmatrix}^l_{p0+t+1} \tag{51}$$

We solve the third equation of Eq. (51) for $(V_j)^l_{p0+t+1}$ and substitute it into Eq. (51) which is reduced to

$$\begin{Bmatrix} V_j \\ S_j \end{Bmatrix}^r_{p0+p} = \begin{bmatrix} b_{21} - \dfrac{b_{22}b_{31}}{b_{32}}, & b_{23} - \dfrac{b_{22}b_{33}}{b_{32}} \\ b_{41} - \dfrac{b_{42}b_{31}}{b_{32}}, & b_{43} - \dfrac{b_{42}b_{33}}{b_{32}} \end{bmatrix} \begin{Bmatrix} V_i \\ S_i \end{Bmatrix}^l_{p0+t+1}$$

$$= [B] \begin{Bmatrix} V_i \\ S_i \end{Bmatrix}_{p0+t+1} = [B] \begin{bmatrix} 1 & \delta_i \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} V_i \\ S_i \end{Bmatrix}^r_{p0+t+1} \tag{52}$$

Combining Eq. (49) and (52) we have

$$\begin{Bmatrix} V_j \\ S_j \end{Bmatrix}^r_{p0+p} = [B] \begin{bmatrix} 1 & \delta_i \\ 0 & 1 \end{bmatrix} [A] \begin{bmatrix} 1 & \delta_j \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} V_i \\ S_i \end{Bmatrix}^r_{p0}$$

$$= [C] \begin{Bmatrix} V_i \\ S_i \end{Bmatrix}^r_{p0} \tag{53}$$

where

$$[C] = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

The frequency equation is

$$c_{11} + c_{22} - 2 = -4\sin mp\pi/N \tag{54}$$

The Packet Consisting of a Finite Number of Blades

Suppose that there are p similar blades connected by welded lacing wires (see Fig. 7). We apply the method of dynamic stiffness to this problem. The equation of equilibrium of forces for blade $i+1$ is

$$\{Q\}^r_{i+1} = \{Q\}^l_{i+1} + [K]\{q\}_{i+1} \tag{55}$$

Equation (12) for lacing wires may be solved and rewritten in the following form:

$$\begin{aligned} \{Q\}^l_{i+1} &= [G]\{q\}_i + [H]\{q\}_{i+1} \\ \{Q\}^r_i &= [E]\{q\}_i + [F]\{q\}_{i+1} \end{aligned} \tag{56}$$

Substituting Eq. (56) into Eq. (55), we have

$$\begin{aligned} \{Q\}^r_{i+1} &= [G]\{q\}_i + ([H]+[K])\{q\}_{i+1} \\ \{Q\}^r_i &= [E]\{q\}_i + [F]\{q\}_{i+1} \end{aligned} \tag{57}$$

The end conditions for a finite packet are that the external forces at the extreme blades be equal to zero

$$\{Q\}^l_1 = \{0\}; \quad \{Q\}^r_p = \{0\} \tag{58}$$

Utilizing these end conditions, we can obtain a recurrent relationship between displacements and forces as follows

$$\begin{aligned} &\{Q\}^r_i = [Z_i]\{q\}_i \\ &[Z_i] = [G][Z_{i-1} - E]^{-1}[F] + [H] + [K] \end{aligned} \tag{59}$$

For the last blade

$$\{Q\}^r_p = [Z_p]\{q\}_p = \{0\} \tag{60}$$

The coefficient determinant must equal to zero, thus we obtain the frequency determinant

$$|Z_p| = 0 \tag{61}$$

Conditions of Resonance for a Circumferentially Closed Structure of Blades

Derivation

In actual turbomachinery, there are unavoidable fluctuations of pressure on the circumference of the flow channel. When the disk rotates, blades on the disk will be exposed to time-periodic forces owing to the fluctuation of the pressure. These time-periodic forces may be resolved into harmonics whose frequencies are integer multiples of the angular velocity ω (in rad/s) of the disk. Let us examine one of these harmonics, say the kth harmonic of the exciting forces which can be expressed as follows

$$P_k(\theta,t) = P_k \sin k(\omega t + \theta) \tag{62}$$

where θ is the angle on the disk measured from a reference radius.

The vibration mode of a closed structure of blades with m nodal diameters corresponding to natural frequency ω_m may be expressed as follows:

$$X_m(\theta,t) = -A_m \cos(\omega_m t + m\theta) \tag{63}$$

The condition of resonance of the blades is that the exciting forces are able to do positive work and then to transmit energy to the blades. The work done by exciting forces on vibrational blades in one period is determined by

$$\begin{aligned} W &= \int_0^{2\pi}\int_0^T P_k(\theta,t)\frac{\partial}{\partial t}X_m(\theta,t)\,\frac{N}{2\pi}\,dtd\theta \\ &= \int_0^{2\pi}\int_0^T P_k \sin k(\omega t+\theta)\,\omega_m A_m \sin(\omega_m t + m\theta)\,\frac{N}{2\pi}\,dtd\theta \\ &= \begin{cases} \pi N P_k A_m & (\text{for } m=k \text{ and } \omega_m = k\omega) \\ 0 & (\text{for } m \neq k \text{ or } \omega_m \neq k\omega) \end{cases} \end{aligned} \tag{64}$$

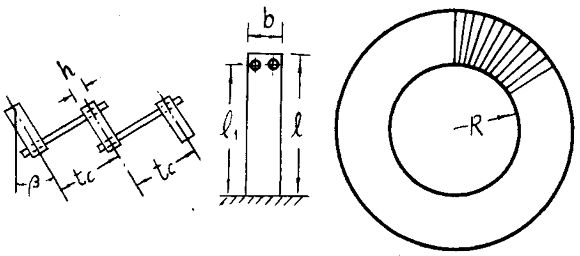

Fig. 8 Rotating bladed disk model.

Table 1 Parameters of rotating model

Radius of disk	$R = 200$ mm
Height of blade	$l = 250$ mm, $l_l = 240$ mm
Width of blade	$b = 40$ mm
Thickness of blade	$h = 4$ mm
Number of blades	$N = 60$
Setting angle	$\beta = 30$ deg
Pitch of blades	$t_c = 39.9$ mm
Type of connection	Z-type loose wire
Angular velocity of disk	$\omega = (15 \sim 25) \times 2\pi$ rad/s

Hence we see that the condition of resonance for a circumferentially closed structure of blades consists of two points:

1) The natural frequency ω_m of the structure with the mode pattern of m nodal diameters should equal the frequency $k\omega$ of the kth harmonic of exciting force, that is,

$$\omega_m = k\omega \tag{65}$$

2) The number of nodal diameters m should coincide with the number of harmonics k, that is

$$m = k \tag{66}$$

In brief:

$$\omega_m / \omega = k = m \tag{67}$$

As these three quantities must coincide with each other, hence this condition of resonance may be called "the triple-coincidence condition." The condition of avoiding triple coincidence may be suggested as a vibration design criterion for blades of this type.

The above conclusion is based upon the assumption that all blades and connecting elements have similar mechanical properties so that the structure is a periodic one. But actually the natural frequencies of blades on a disk usually fluctuate around a mean value. Hence the energy can be transmitted to blades with conditions $\omega_m = k\omega$ and $m \neq k$. But since the values of fluctuation of frequencies are usually small, the vibrations of blades with conditions $\omega_m = k\omega$ but $m \neq k$ are very weak. A vigorous vibration of blades can occur only in the case of the triple-coincidence condition.

Experiment on a Rotating Model

To verify the above conclusion, an experiment on a rotating bladed disk model was performed. The model was specially made for this purpose. The structure of the model is shown in Fig. 8 and the parameters of the model in Table 1.

The results of experiments are shown in Fig. 9. Because the angular velocity of the model disk is relatively low, and the natural frequency of the blades is relatively high, we can get resonance only in the case of $k = 10 \sim 15$. It shows that the calculated results coincide with the measured ones.

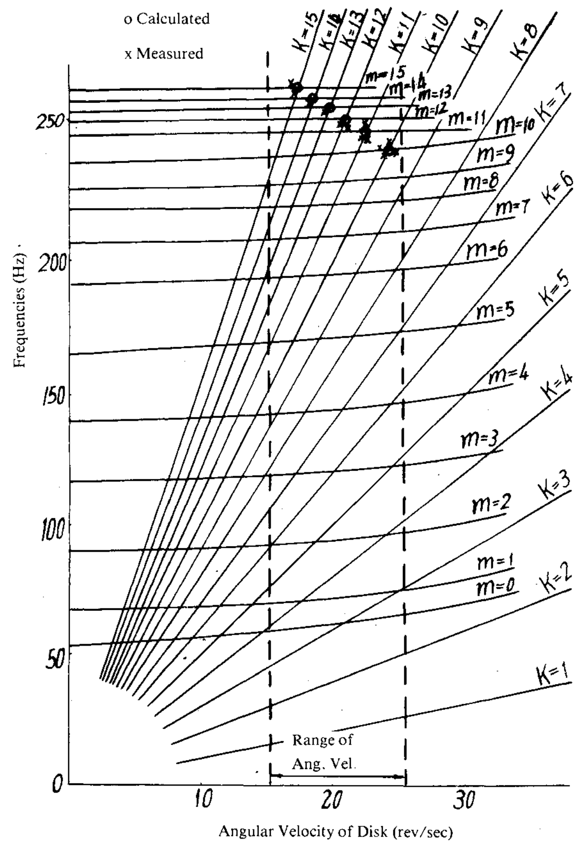

Fig. 9 Campbell diagram of model disk showing calculated and experimental results.

Conclusion

The method presented in this paper for calculating frequencies and mode shapes of circumferentially closed and unclosed structures of blades can be adopted in turbine design. The condition of resonance of a closed structure of blades on a disk under time-periodic excitation analyzed in this paper has been verified by experiments. The criterion suggested for vibration design of blades is proved to be reasonable for turbine design.

Acknowledgment

The author gratefully acknowledges the assistance of Deng Lian-chao who developed the computer program, Zhao Yu-chang who performed the calculations, and Tang Bao-Kang, Xu He-nian, Gai Wei, and Guo Yue-mei who performed the experiments.

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